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ON THE EXISTENCE OF NON-STATIC PLANE SYMMETRIC SPACE-TIMES

by

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I. INTRODUCTION

In the general theory of relativity, when the metric tensor obtained as a solution of Einstein's field equations for empty space depends on the time coordinate, it is often said to show the existence of pure gravitational waves filling the region of space-time considered. Similarly, in the case of the Einstein-Maxwell system of equations of relativistic electrodynamics when a certain solution depends on the time coordinate, it may represent electromagnetic waves or coupled electromagnetic and gravitational waves. Since in problems of general relativity due to the principle of covariance, a wide range of coordinate systems are admissible equally well, it is not surprising if in some instances a specific coordinate system may be introduced so that in the new system of coordinates, the solution becomes independent of time. For a rigourous justification of the wave motion or the true time dependence of the solution, it should be shown that the the space-time represented by the solution is not static. A static space-time in general relativity is defined in an invariant manner through the requirement: that the space-time admits a global continuous time-like group of motions carrying a region of space-time into itself. Professor Taub[1] has introduced the notion of plane symmetry and the group of transformations that define it and has shown that plane symmetric empty space-time admits a one-parametric, continuous, time-like group of motions. Thus, there exists no plane gravitational waves filling a region of space-time. A similar result holds in the case of spherical symmetry, as is well known from Birkhoff's theorem[2], that space-times with spherical symmetry satisfying Einstein's field equations for empty space reduce to that of the Scwarzschild's static solution. Further, it is known that in the case of the Einstein-Maxwell system of equations, when no charges are present, solutions with spherical symmetry are necessarily static. Thus, there exists no spherical symmetric gravitational or electromagnetic waves.

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1. A. H. Taub, "Empty Space-Times Admitting a Three Parametric Group of Motions",
        Annals of Mathematics <u>53</u>, 1951, pp. 472-490

2. R. C. Tolman, <u>Relativity, Thermodynamics and Cosmology</u>, Oxford, 1934, p. 252

The analogous problem in the case of plane symmetry is of interest, and it is worth investigating whether there exists plane electromagnetic waves in general relativity. Recently, Takeno[3] has obtained one particular solution of the Einstein-Maxwell equations in which the metric tensor exhibits plane symmetry. This solution depends on the time coordinate. It is the purpose of this paper to examine whether this time dependence can be removed by an admissible coordinate transformation, in other words, whether the space-time given in Takeno's solution admits a continuous global one parameter time-like group of motions. It will be shown that this will be the case only if the space-time is flat, in which case, the space-time admits the 10 parameter Lorentz group if the coordinate system used is taken to be an inertial one. It may be remarked here that the electromagnetic field tensor in Takeno's solution is not truly plane symmetric in the sense that it does not remain invariant under the group of transformations that define plane symmetry. The problem of obtaining the general solution when both the metric tensor and Maxwell's field tensor are plane symmetric and the determination of whether that solution admits a continuous global group of motions containing a time-like generator remains to be done. This will then answer the question of the existence of plane electromagnetic waves in general relativity.

II. KILLING'S EQUATIONS

The necessary and sufficient condition that a given space-time is static, or in other words, the space-time admits a global continuous group of motions carrying a region of space-time into itself containing a time-like generator is given by the following:

There exists a contravariant vector ξ^α obtained as a solution of the system of linear partial differential equations, known as the Killing equations[4], viz.

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0, \qquad\qquad \alpha, \beta = 1, 2, 3, 4 \qquad (2.1)$$

satisfy the condition

$$g_{\alpha\beta}\, \xi^\alpha\, \xi^\beta > 0 \qquad\qquad\qquad (2.2)$$

3. H. Takeno,"On Plane Wave Solutions of Field Equations in General Relativity", Tensor 7, 1957, pp. 97-102

4. L. P. Eishenhart, Riemannian Geometry, Princeton, 1949, Chapter VI

(where a semicolon denotes covariant differentiation with respect to the given metric field and in what follows a comma will denote partial differentiation with respect to the variable indicated). The equations (2.1) could be written in terms of the contravariant vector ξ^α as

$$\xi^\sigma g_{\alpha\beta,\sigma} + g_{\sigma\beta}\xi^\sigma_{,\alpha} + g_{\sigma\alpha}\xi^\sigma_{,\beta} = 0 \ . \tag{2.1}'$$

As it is well known, these equations express the fact that the metric tensor $g_{\alpha\beta}$ remains invariant under the infinitesimal transformation

$$x^{*\rho} = x^\rho + \xi^\rho(x^\alpha) \ . \tag{2.3}$$

It is easy to see that if a space-time admits a group whose infinitesimal generators are given by $\xi^i = 0$ ($i = 1, 2, 3$), $\xi^4 \neq 0$, then it is static, i.e., $g_{\alpha\beta,4} = 0$. It is known that the maximum number of the parameters of the group of motions for a given space-time is 10 and that is attained for a space of constant curvature.

If instead of the symmetric tensor $g_{\alpha\beta}$, the antisymmetric Maxwell tensor $F_{\alpha\beta}$ is considered, then the invariance of $F_{\alpha\beta}$ under the infinitesimal transformation (2.3) gives rise to the system of equations

$$\xi^\sigma F_{ij,\sigma} + F_{i\beta}\xi^\beta_{,j} + F_{\alpha j}\xi^\alpha_{,i} = 0 \ . \tag{2.4}$$

The solutions of these equations for a given field tensor $F_{\alpha\beta}$ determine the symmetry properties of the field or the group of motions which the electromagnetic field admits. Conversely, for a given group of transformations, equations (2.1) and (2.4) determine certain general properties of the metric tenson $g_{\alpha\beta}$ and the electromagnetic field tensor $F_{\alpha\beta}$, respectively, that remain invariant under that group.

III. ELECTROMAGNETIC FIELDS WITH PLANE SYMMETRY IN GENERAL RELATIVITY

The system of Einstine-Maxwell equations of relativistic electrodynamics is given by (in the absence of charge or current):

$$R^{\mu}_{\nu} - \frac{1}{2} g^{\mu}_{\nu} R = -kc^2 T^{\mu}_{\nu} \quad , \quad k = \frac{8\pi G}{c^2}$$

$$T^{\mu}_{\nu} = F^{\mu\sigma} F_{\nu\sigma} - \frac{1}{4} g^{\mu}_{\nu} F_{\rho\sigma} F^{\rho\sigma} \tag{3.1}$$

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0$$

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} (F^{\mu\nu} \sqrt{-g}) = 0$$

(where, as usual, Greek letters run from 1 through 4). The last two in the set
of Equations (3.1) are the eight Maxwell equations, and if a solution of these
Maxwell equations for the field tensor $F_{\alpha\beta}$ could be obtained in terms of the
coordinates x^{α} and the presupposed metric tensor $g_{\alpha\beta}$, then the second of the
equations in the set (3.1) determines the energy tensor T^{μ}_{ν}. This energy tensor
T^{μ}_{ν} will satisfy the divergence equations $T^{\mu}_{\nu;\mu} = 0$, and thus the Einstein equa-
tions, first, in the set of equations (3.1) should be integrable. In this way,
a solution for $g_{\alpha\beta}$, $F_{\alpha\beta}$ of the system (3.1) could be obtained. It has been shown
by Taub[5] that space-times with plane symmetry should remain invariant under the
group of transformations

$$x^{2*} = x^2 \cos \Theta + x^3 \sin \Theta + a$$

$$x^{3*} = -x^2 \sin \Theta + x^3 \cos \Theta + b \tag{3.2}$$

$$x^{1*} = x^1, \qquad x^{4*} = x^4 \quad , \qquad a, b, \Theta \text{ arbitrary constants}$$

and the line element for such space-times reduces to:

$$ds^2 = -B(dx^2 - dt^2) - A(dy^2 + dz^2) \tag{3.3}$$

A, B are functions of x and t only, $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = t$.
It can be shown from the equations (2.4) with the ξ^{α}_{a} ($\alpha = 1, 2, 3, 4$; a =
1, 2, 3) determined from equations (3.2) that for an electromagnetic field tensor
$F_{\alpha\beta}$ with plane symmetry one should have

$$F_{12} = 0, \quad F_{13} = 0, \quad F_{24} = 0, \quad F_{34} = 0 .$$

5. op. cit., A. H. Taub

So the only non-vanishing components of the Maxwell tensor exhibiting plane symmetry are F_{14}, F_{23}; the former gives the component of the electric field and the latter of the magnetic field. Solutions in plane symmetry with electromagnetic fields analogous to Reissner-Nordström[6] solution in the spherical symmetric case exist, and such solutions are discussed in the appendix.

IV. TAKENO'S SOLUTION

In obtaining his solution Takeno assumes that the functions A and B entering in the expression for the line element (3.3) with plane symmetry depends only on the combination $u = x-t$. He takes at start $F_{14} = 0$, $F_{23} = 0$. Then, he obtains the following solution of Einstein-Maxwell system of equations (3.1):

$$A = A(u), \quad B = B(u), \quad u = x-t, \quad \rho = \rho(u), \quad \sigma = \sigma(u)$$

$$g_{11} = -g_{44} = -B, \quad g_{22} = g_{33} = -A, \quad g_{\alpha\beta} = 0 \quad (\alpha \neq \beta) \quad (4.1)$$

$$F_{34} = F_{13} = \sigma, \quad F_{21} = F_{42} = \rho, \quad F_{14} = 0, \quad F_{23} = 0$$

where A, B, ρ, σ are related through the single equation

$$A'' - \frac{A'^2}{2A} - \frac{A'B'}{B} = -8\pi k(\rho^2 + \sigma^2), \quad \text{where } k = \frac{8\pi G}{c^2}. \quad (4.2)$$

The condition that the space-time is flat is given by

$$A'' - \frac{A'^2}{2A} - \frac{A'B'}{B} = 0 \quad (4.3)$$

where a prime denotes differentiation with respect to the argument of the function. So, the vanishing of ρ and σ implies the space-time is flat and conversely. It may be further remarked that for the space-time represented by the solution (4.1) $R_{\mu\nu} = 0$ implies $R^{\alpha}_{\beta\gamma\delta} = 0$. From the solution, it is evident that there is considerable arbitrariness in it, making the physical meaning rather unclear. As has been shown earlier, the electromagnetic field tensor is plane symmetric only if all its components, except F_{14} and F_{23}, vanish. Thus, the electromagnetic field does not show plane symmetry in this solution. Under the transformation

$$u = x-t, \quad v = x+t, \quad x^{2'} = x^2, \quad x^{3'} = x^3, \quad \text{the line element}$$

6. Sir Arthur Eddington, <u>The Mathematical Theory of Relativity</u>, Cambridge, 1930, p. 185

$$ds^2 = -B(dx^2 - dt^2) - A(dy^2 + dz^2), \quad B = B(u), \quad A = A(u)$$

takes the form

$$ds^2 = -B(u)\,dudv - A(u)\left[(dx^{2'})^2 + (dx^{3'})^2\right]$$

Introducing now new variables U, V, X^2, X^3, defined through the equations

$$dU = B(u)du, \quad \text{i.e.,} \quad U = \int B(u)du + c, \quad c \text{ an arbitrary constant}$$

$$V = v, \quad X^2 = x^{2'}, \quad X^3 = x^{3'}$$

the line element given above assumes the form

$$ds^2 = -dUdV - A(U)\left[(dX^2)^2 + (dX^3)^2\right]$$

which now could be written as

$$ds^2 = -(dx^2 - dt^2) - A(u)(dy^2 + dz^2), \quad u = x-t, \tag{4.4}$$

just by relabeling the old system of coordinates. So there is no loss of generality by treating the line-element (4.4) instead of the line-element (3.3) for the present discussion. The necessary and sufficient condition that the line-element given by (4.4) is flat is given by

$$\frac{A''}{2} - \frac{A'^2}{4A} = 0 \quad . \tag{4.5}$$

Rewriting this equation as

$$\frac{A''}{A'} - \frac{1}{2}\frac{A'}{A} = 0 \tag{4.5'}$$

and performing the two integrations, the solution for A is given by

$$A = (Cu-D)^2 \tag{4.6}$$

where C, D are constants of integration. So, in order that the space-time given by (4.4) be flat, A must satisfy this equation, and conversely. The case when the constant C vanishes gives flat space-time in an inertial coordinate system. When C is non-vanishing, the coordinate system is curvilinear, but the metric is still flat.

V. SOLUTION OF KILLING EQUATIONS

The system of 10 Killing equations $(2.1)'$ for a metric tensor $g_{\alpha\beta}$ given by the line element (4.4) is given by the following:

$$\xi^1_{,1} = 0 \qquad\qquad (1:1)$$

$$\xi^4_{,4} = 0 \qquad\qquad (4:4)$$

$$\xi^1_{,4} = \xi^4_{,1} \qquad\qquad (1:4)$$

$$A\,\xi^2_{,1} + \xi^1_{,2} = 0 \qquad\qquad (1:2)$$

$$A\,\xi^3_{,1} + \xi^1_{,3} = 0 \qquad\qquad (1:3)$$

$$\xi^2_{,3} + \xi^3_{,2} = 0 \qquad\qquad (2:3)$$

$$A\,\xi^2_{,4} - \xi^4_{,2} = 0 \qquad\qquad (2:4)$$

$$A\,\xi^3_{,4} - \xi^4_{,3} = 0 \qquad\qquad (3:4)$$

$$\xi^2_{,2} = \frac{1}{2}\frac{A'}{A}\,(\xi^4 - \xi^1) \qquad\qquad (2:2)$$

$$\xi^3_{,3} = \frac{1}{2}\frac{A'}{A}\,(\xi^4 - \xi^1) \qquad\qquad (3:3)$$

$$(5.1)$$

Introducing the new variables
$$u = x^1 - x^4 , \qquad v = x^1 + x^4$$
and observing that a function F of x^1, x^4 becomes a function of u and v with respect to the new variables, we have, by the rules of partial differentiation,

$$F_{,u} = \frac{1}{2}(F_{,1} - F_{,4}) \qquad\qquad F_{,1} = F_{,u} + F_{,v}$$

$$F_{,v} = \frac{1}{2}(F_{,1} + F_{,4}) \qquad\qquad F_{,4} = F_{,v} - F_{,u}$$

$$(5.2)$$

Equations (1:2), (1:3), (2:4), (3:4) of the set (5.1) take the following form when the transformation to the new variables u and v is performed:

$$2A \, \xi^2{}_{,v} + (\, \xi^1 - \xi^4 \,)_{,2} = 0$$

$$2A \, \xi^2{}_{,u} + (\, \xi^1 + \xi^4 \,)_{,2} = 0$$

$$2A \, \xi^3{}_{,v} + (\, \xi^1 - \xi^4 \,)_{,3} = 0 \qquad (5.3)$$

$$2A \, \xi^3{}_{,u} + (\, \xi^1 + \xi^4 \,)_{,3} = 0$$

It is understood that in this transformation the components of the contravariant vector ξ^α are now functions of the variables u, v, x^2 and x^3. Equations (1:1), (1:4), (4:4) of the set (5.1) now give on partial differentiation the following equations:

$$\xi^1{}_{,41} = \xi^1{}_{,14} = 0, \qquad\qquad \xi^1{}_{,44} = 0$$

$$\qquad (5.4)$$

$$\xi^4{}_{,41} = \xi^4{}_{,14} = 0, \qquad\qquad \xi^4{}_{,11} = 0 \quad .$$

The last of these equations gives on integration,

$$\xi^4{}_{,1} = e(x^2, x^3, x^4)$$

where e is an arbitrary function of the variables indicated. Because of the equation $\xi^4{}_{,14} = 0$, the expression for $\xi^4{}_{,1}$ reduces to $\xi^4{}_{,1} = e(x^3, x^3)$ which gives on integration,

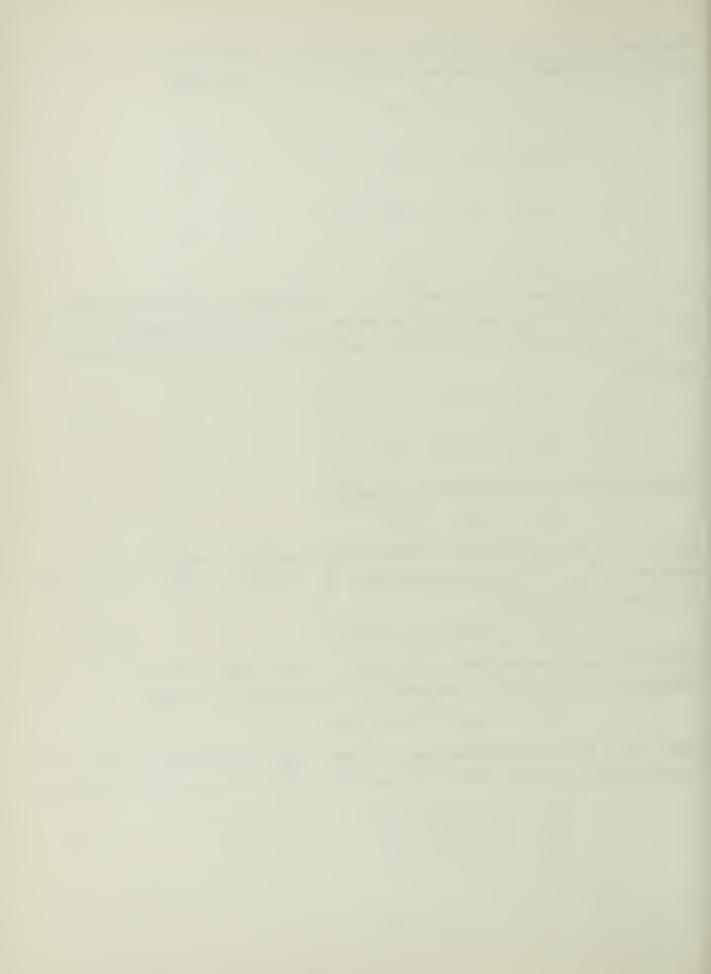$$\xi^4 = x^1 \, \acute{e}(x^2, x^3) + h(x^2, x^3) \qquad (5.5)$$

where both e and h are arbitrary functions of x^2 and x^3 only. Similarly, on integrating the first pair of the equations in equation (5.4), we have

$$\xi^1 = x^4 \, g(x^2, x^3) + f(x^2, x^3) \qquad (5.6)$$

where g and f are arbitrary functions of x^2 and x^3 only. Equations (5.5) and (5.6), with the help of equation (1:4) of the set (5.1), give e = g. So, we have now:

$$\xi^1 - \xi^4 = -ue + f - h$$

$$\qquad (5.7)$$

$$\xi^1 + \xi^4 = ve + f + h \quad .$$

From the first pair of equations in (5.3), we have by partial differentiation with respect to u and v, respectively,

$$\xi^2_{,vu} + \left(\frac{1}{2A}\right)' \left[-ue_{,2} + (f-h)_{,2}\right] - \frac{1}{2A}e_{,2} = 0$$

(5.8)

$$\xi^2_{,vu} + \frac{1}{2A}e_{,2} = 0$$

where $A = A(u)$, so $A_{,v} = 0$. From equations (5.8) it follows that because

$$\xi^2_{,uv} = \xi^2_{,vu} \quad ,$$

$$e_{,2}\left[-u\left(\frac{1}{2A}\right)' - \frac{1}{A}\right] + \left(\frac{1}{2A}\right)' (f-h)_{,2} = 0$$

i.e.,

$$\frac{e_{,2}}{(f-h)_{,2}} = \frac{\left(\frac{1}{2A}\right)'}{u\left(\frac{1}{2A}\right)' + \frac{1}{A}} \quad .$$

We observe that the left-hand side of this equation is an arbitrary function of the variables x^2 and x^3, while the right-hand side may be taken as a known function of the variable $u = x^1 - x^4$ when $A(u)$ is given. It is clear that this equation could be satisfied only if each side equals to some constant. Thus, equating each side of the last equation to a constant k, we have
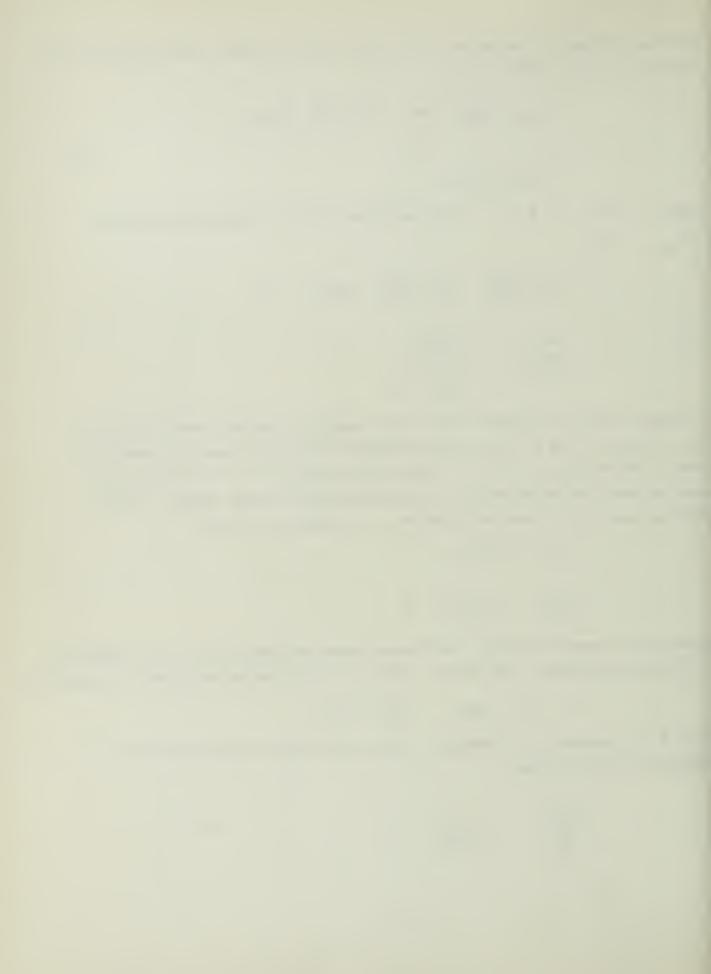
$$e_{,2} = k(f-h)_{,2}$$

$$\left(\frac{1}{2A}\right)' = ku\left(\frac{1}{2A}\right)' + \frac{k}{A} \quad .$$

These are two among the various consistency equations that arise in the integration of Killing's equations. The second in the last pair of equations gives on integration,

$$A = (\bar{c} - k\bar{c}u)^2 = (\bar{c})^2 (1-ku)^2$$

where $\bar{c}$ is a constant of integration. This is evident by rewriting the last equation as the following:

$$\frac{\left(\frac{1}{2A}\right)'}{\frac{1}{2A}} = \frac{2k}{1-ku} \quad .$$

The solution for A as a function of u as obtained from the last equation, that is,

$$A = (\bar{c} - k\bar{c}u)^2$$

is exactly the condition that the space-time be flat, equation (4.6), on identifying the constants as $\bar{c} = -D$, $k\bar{c} = -C$. The problem now gives rise to three different cases which should be treated separately.

Case (i). $A = \text{constant}$, $e_{,2} = 0$, f and h are so far arbitrary functions of x^2 and x^3 only. The metric of the space-time reduces to that of special relativity in an inertial coordinate system.
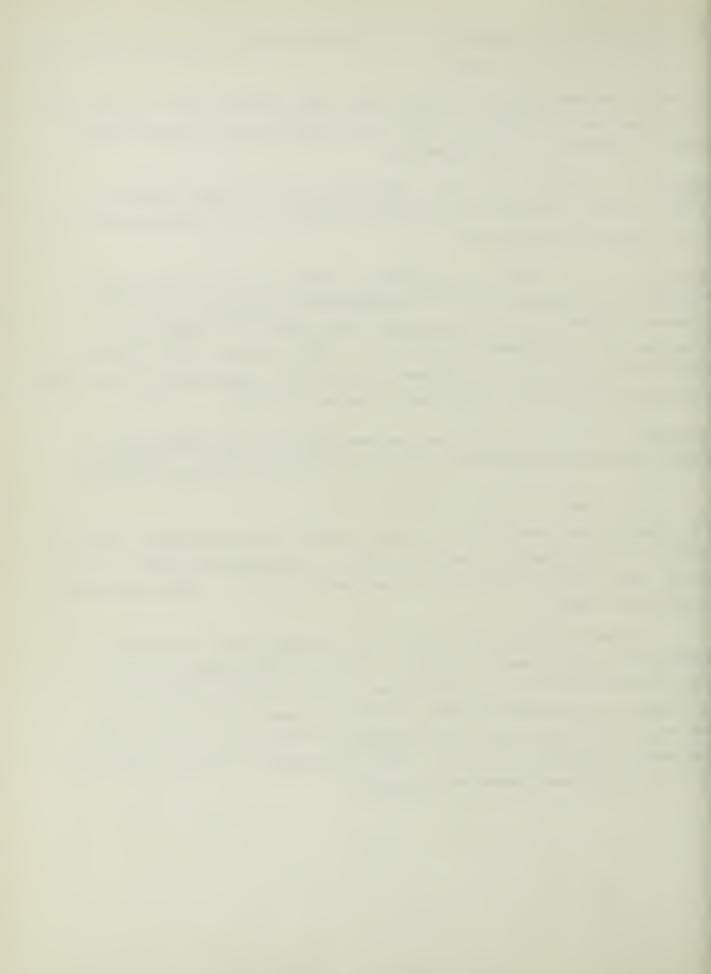
Case (ii). $A = (\bar{c} - k\bar{c}u)^2$, $e_{,2} = k(f-h)_{,2}$ where e, f, and h are arbitrary functions of the variables x^2 and x^3 satisfying the last equation, and k is a constant. The metric $g_{\alpha\beta}$ of the space-time then satisfies the condition for being flat, i.e., the Riemann-Christófell tensor $R^{\alpha}_{\beta\gamma\delta}$ vanishes. The coordinate system used is not inertial in the sense that the metric components $g_{\alpha\beta}$ do not reduce to constants as in the metric usually used in special relativity.

Case (iii). A is not any of the functions mentioned in earlier cases, but, in general, may take any functional value of u. But we now have the two equations:

$$e_{,2} = 0, \qquad (f-h)_{,2} = 0 \ .$$

This situation now corresponds to one where the metric represents curved space-time; neither all the components of the Ricci tensor nor the Riemann-Christofell tensor vanish. Thus, the field represented by the space-time is due to some non-vanishing stress energy tensor.

Takeno's solution corresponds to the case where there is an electro-magnetic field in the absence of charges. Before proceeding further in this case of a curved metric, it is observed that Killing's equations (5.1) give as their solutions ten parametric groups of motions in cases (i) and (ii). In the case of (i), as is well known, the ten parametric Lorentz group of four transla-tions and six rotations is obtained. In fact, equations (5.1) admit, in this case, the Lorentz group whose generators are given by

$$p_i \qquad\qquad i = 1, 2, 3, 4$$

$$x_1 p_2 - x_2 p_1 \quad, \qquad x_2 p_3 - x_3 p_2 \quad, \qquad x_3 p_1 - x_1 p_3$$

$$x_1 p_4 + x_4 p_1 \quad, \qquad x_2 p_4 + x_4 p_2 \quad, \qquad x_3 p_4 + x_4 p_3 \quad.$$

The generators of the group are defined through the equations

$$\underset{a}{X} f = \underset{a}{\xi^i} \frac{\partial f}{\partial x^i}$$

$$p_i = \frac{\partial f}{\partial x^i}$$

where $i = 1, 2, 3, 4,$ $a \leq 1, 2, \ldots, 10.$

In the case of (ii) there is obtained the ten parametric groups of motions generated by infinitesimal transformations that keep the distance function
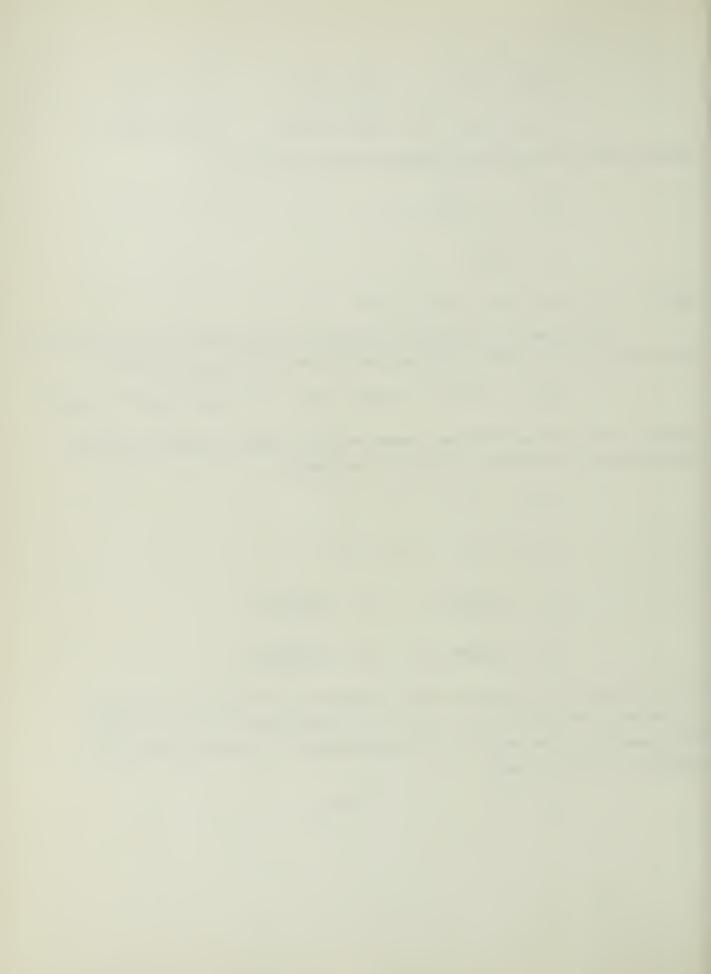
$$ds^2 = -\left[(dx^1)^2 - (dx^4)^2\right] - \left[C - (x^1 - x^4)D\right]^2 \left[(dx^2)^2 + (dx^3)^2\right]$$

invariant. The system of Killing's equations (5.1) admits through its solutions the group whose infinitesimal generators are given by

$$p_2, \ p_3, \qquad p_1 + p_4, \qquad p_1 - p_4$$

$$x^2 p_3 - x^3 p_2, \qquad x^1 p_4 + x^4 p_1$$

$$x^3 p_1 + \frac{1}{C(Cu-D)} p_3, \qquad x^2 p_1 + \frac{1}{C(Cu-D)} p_2$$

$$x^3 p_4 + \frac{1}{C(Cu-D)} p_3, \qquad x^2 p_4 + \frac{1}{C(Cu-D)} p_2 \quad .$$

Now, returning to the situation where $A \neq$ constant, $A \neq (Cu-D)^2$, but otherwise A is an arbitrary function of $u = x-t$. It has been shown that from the first three equations of the set (5.1) and the consistency of the first pair of the equations in (5.3), we have

$$e_{,2} = 0, \qquad\qquad (f-h)_{,2} = 0 \quad .$$

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The consistency of the remaining two equations in the set (5.3) gives rise to

$$e_{,3} = 0, \qquad\qquad (f-h)_{,3} = 0.$$

This could be seen in an exactly similar manner. It may be noted that equations (5.3) replace four of the ten Killing equations and three of the latter set have already been integrated. Remembering that e, f, and h are functions of the variables x^2 and x^3 alone, we have

$$e = \text{constant, } k_1 \text{, say}$$

$$f-h = \text{constant, } k_2 \text{, say}$$

So, we have

$$f = \alpha(x^2, x^3)$$

$$e = k_1$$

$$h = \alpha(x^2, x^3) + k_2$$

Finally, expressions for ξ^1, ξ^4 reduce to the following:

$$\xi^1 = k_1 x^4 + \alpha(x^2, x^3)$$
$$\xi^4 = k_1 x^1 + \alpha(x^2, x^3) + k_2$$

(5.9)

where k_1, k_2 are arbitrary constants and α, so far, is an arbitrary function of x^2 and x^3 only. The remaining equations of the set (5.1) to be satisfied when rewritten in terms of the new variables $u = x^1 - x^4$, $v = x^1 + x^4$, give rise to the following equations:

$$2A \; \xi^2_{,v} = 0$$

$$2A \; \xi^3_{,v} = 0$$

$$A \; \xi^2_{,u} + \alpha_{,2} = 0$$

$$A \; \xi^3_{,u} + \alpha_{,3} = 0$$

(5.10)

$$\xi^2_{,3} + \xi^3_{,2} = 0$$

$$\xi^2_{,2} = \xi^3_{,3} = \frac{1}{2} \frac{A'}{A}(k_1 u + k_2)$$

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Solutions for ξ^2 and ξ^3 in the first two of this set of equations state that they are independent of v. By integrating, we have from the last pair of equations

$$\xi^2 = \frac{1}{2}\left(\frac{A'}{A}\right)(k_1 u + k_2)\, x^2 + L(u,\, x^3)$$

$$\xi^3 = \frac{1}{2}\left(\frac{A'}{A}\right)(k_1 u + k_2)\, x^3 + M(u,\, x^3)$$

(5.11)

where $A = A(u)$; L and M are arbitrary functions of the variables indicated; k_1 and k_2 are arbitrary constants.

From the fifth equation in the set (5.10), we obtain

$$L(u,\, x^3) = P(u)\, x^3 + Q(u)$$

$$M(u,\, x^2) = -P(u)\, x^2 + R(u)$$

where P, Q, R are arbitrary functions of the variable u only. At this stage we have all the ten Killing equations (5.1) solved except the following two:

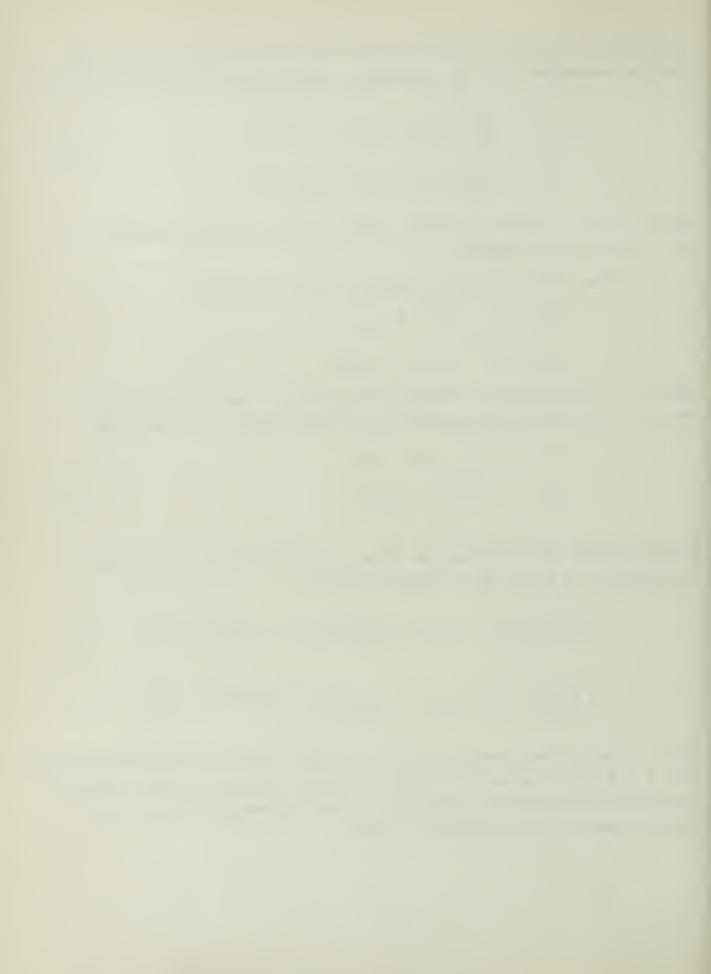$$\xi^2_{,u} = -\frac{1}{A}\, \alpha_{,2}\, (x^2,\, x^3)$$

$$\xi^3_{,u} = -\frac{1}{A}\, \alpha_{,3}\, (x^2,\, x^3)\quad .$$

(5.12)

Now substituting the expressions for ξ^2 and ξ^3 given by (5.11) in the last two equations, we obtain the following conditions:

$$A\left[\frac{1}{2}\left(\frac{A'}{A}\right)'(k_1 u + k_2)\, x^2 + \frac{1}{2}\left(\frac{A'}{A}\right)k_1 x^2 + P'(u)x^3 + Q'(u)\right] = -\alpha_{,2}$$

$$A\left[\frac{1}{2}\left(\frac{A'}{A}\right)'(k_1 u + k_2)\, x^3 + \frac{1}{2}\left(\frac{A'}{A}\right)k_1 x^3 - P'(u)x^2 + R'(u)\right] = -\alpha_{,3}$$

where α is an arbitrary function of x^2 and x^3, and P, Q, R are arbitrary functions of u only, A is a given function of u with the exceptions mentioned earlier, that gives the condition for space-time becoming flat. The consistency of the last two equations demand, as can easily be seen, that

$$P'(u) = 0, \quad \text{i.e.,} \quad P = \text{constant} = k_3, \text{ say}$$

Then, the last two equations may be written as:

$$A\left[x^2\left(\frac{1}{2}\left(\frac{A'}{A}\right)'(k_1 u + k_2) + \frac{1}{2}\left(\frac{A'}{A}\right)k_1\right) + Q'(u)\right] = -\alpha_{,2}$$

$$A\left[x^3\left(\frac{1}{2}\left(\frac{A'}{A}\right)'(k_1 u + k_2) + \frac{1}{2}\left(\frac{A'}{A}\right)k_1\right) + R'(u)\right] = -\alpha_{,3}$$

(5.13)

where, as noted earlier, α is an arbitrary function of the variables x^2 and x^3 only; Q and R are arbitrary functions of u only; A may be supposed to be a given function of u. We have, from the last two equations on partial differentiation with respect to x^2 and x^3, respectively, the following consistency equations:

$$A\left[\frac{1}{2}\left(\frac{A'}{A}\right)'(k_1 u + k_2) + \frac{1}{2}\left(\frac{A'}{A}\right)k_1\right] = -\alpha_{,22}$$

$$A\left[\frac{1}{2}\left(\frac{A'}{A}\right)'(k_1 u + k_2) + \frac{1}{2}\left(\frac{A'}{A}\right)k_1\right] = -\alpha_{,33} \quad .$$

(5.14)

Thus, it follows that $\alpha_{,22} - \alpha_{,33} = 0$. Further, it is observed that the left-hand sides of these two equations depends on functional values of u only, whereas the right-hand sides are arbitrary functions of x^2 and x^3 only. Thus, each side of the last two equations must be reduced to constants as a necessary condition for the equations to be admissible. It can be seen that this, indeed, will be the case only if

(a) k_1 and k_2 are both zero; A an arbitrary function of u.

or (b) A = constant, or A = $(Cu-D)^2$, where C, D are constants that are related to the constants k_1 and k_2. The latter follows by rewriting (5.14) in the following form

$$k_1\left[\frac{1}{2}\left(\frac{A'}{A}\right)'Au + \frac{1}{2}\left(\frac{A'}{A}\right)A\right] + k_2\left[\frac{1}{2}\left(\frac{A'}{A}\right)'A\right] = -\alpha_{,22}$$

$$= -\alpha_{,33} \quad .$$

For each numerical value of the constants k_1 and k_2, one has a tranformation that belongs to the group of motions. A, being any function of u, the coefficients of k_1 and k_2 may be considered to be arbitrary functions of u and independent of k_1 and k_2. The expressions on the right-hand side of the above equation are independent of u, and, hence, in order for this equation to be admissible, the coefficients of k_1 and k_2 must reduce to constants. The solution to the two differential equations thus obtained give for A, the solution $A = (Cu-D)^2$, where the constants C, D, k_1, k_2 are related through the equations

$$-k_1 CD - k_2 C^2 = -\alpha_{22} = -\alpha_{33} \ .$$

The situation in (b) gives flat space-time, cases (i) and (ii) discussed earlier. Returning then to the situation for a curved space-time, where A may be any function of u (obtained as a solution of the field equations with a non-vanishing stress-energy tensor), corresponding to the case (a) is:

$$k_1 = 0, \quad k_2 = 0, \quad \text{hence,} \quad \alpha_{,22} = 0, \quad \alpha_{,33} = 0,$$

α being a function of x^2 and x^3 only, its solution from the last two equations for $\alpha(x^2, x^3)$ is:
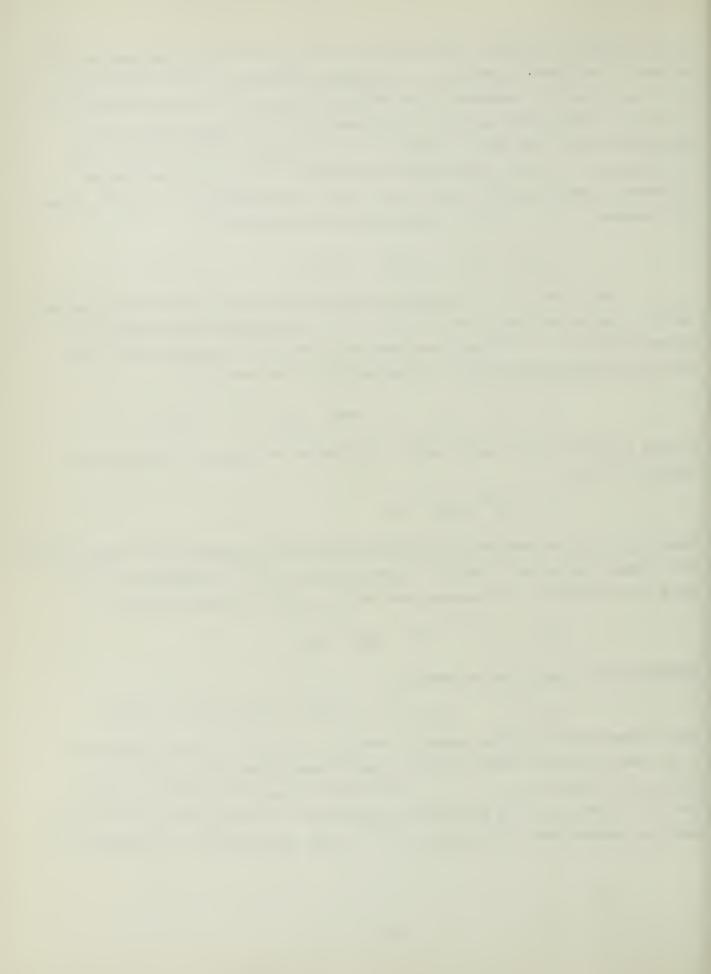
$$\alpha = k_4 x^2 + k_5 x^3 + k_6$$

where k_4, k_5, k_6 are arbitrary constants arising out of integrations. Thus, $\alpha(x^2, x^3)$ is a linear function of x^2 and x^3. Expressions for ξ^1, ξ^4 obtained in (5.9), now give as a result of the expressions for k_1, k_2 and α obtained just now,

$$\xi^1 = \xi^4 = k_4 x^2 + k_5 x^3 + k_6 \ .$$

Considering the sign of the expression

$$g_{\mu\nu} \xi^\mu \xi^\nu = g_{11}[(\xi^1)^2 - (\xi^4)^2] + g_{22}[(\xi^2)^2 + (\xi^3)^2],$$

(for a space-time with plane symmetry) where g_{11} and g_{22} are negative quantities, it is clear that for a space-time with plane symmetry, the Killing vector ξ^α could not be time-like if $\xi^1 = \xi^4$. From what has been shown above, it follows that for a space-time with non-vanishing curvature, the consistency of the Killing equations demands that ξ^1 be equal to ξ^4. Thus, the generators of the group of
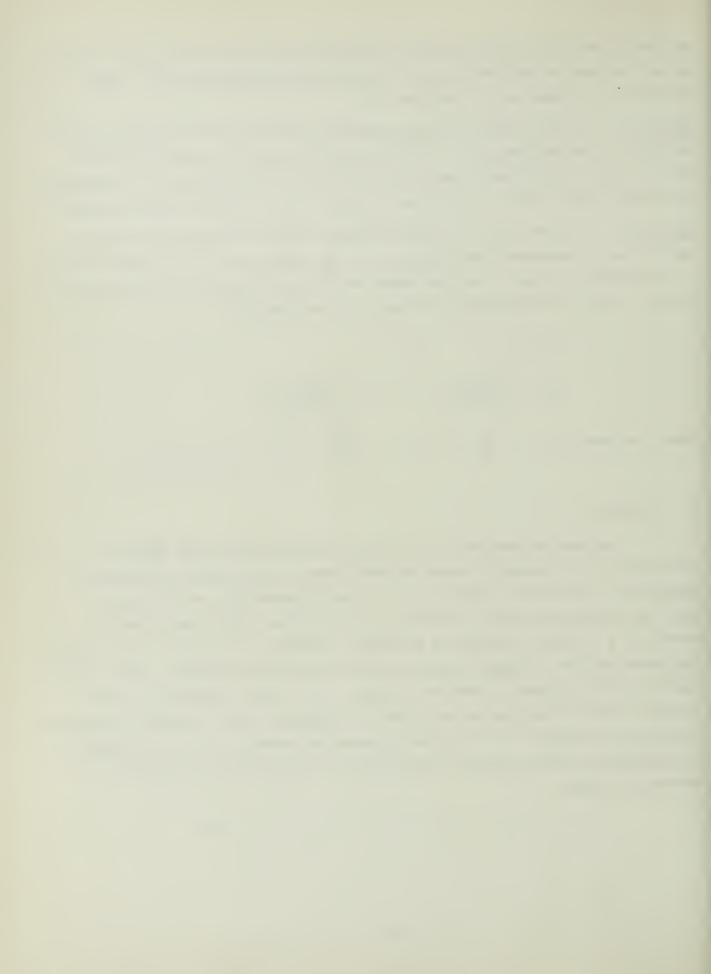
motions in the case of the space-time discussed form a space-like or null vector unless one goes to the limiting case of the space-time becoming flat. These situations are summarized in the form of:

Theorem 1. For space-times with plane symmetry and the metric components depending only on the combination $x^1 - x^4$, there exists a global continuous group of motions whose infinitesimal generators form a time-like vector only if the Riemann-Christofell tensor for the space-time vanishes, i.e., the space-time being flat.

Theorem 2. For space-times with plane symmetry having non-vanishing curvature and the metric components depending only on the combination $x^1 - x^4$, there exists a five-parametric group of motions whose infinitesimal generators are space-like or null vectors. The generators of the group are given by

$$p_2, \ p_3, \qquad\qquad p_1 + p_4$$

$$x^2 p_1 - \left(\int \frac{du}{A(u)} \right) p_2, \qquad x^3 p_1 - \left(\int \frac{du}{A(u)} \right) p_3$$

where, as usual, $\underset{a}{X} f = \underset{a}{\xi^i} p_i, \quad p_i = \dfrac{\partial f}{\partial x^i} \quad (i = 1, 2, 3, 4,$
$a = 1, 2, 3, 4, 5)$

VI.   CONCLUSION

The results obtained in the preceding sections show that Takeno's solution for electromagnetic waves in plane symmetric space-time is non-static according to the invariant definition of a static space-time, as it does not admit any continuous global time-like group of motions. This, then gives an example of a solution representing true waves in general relativity. But, as has been shown earlier, although the metric tensor shows plane symmetry, the electromagnetic field in Takeno's solution does not. So, it still remains to be ascertained whether there exists any plane electromagnetic wave solutions in general relativity or whether, as is the case in spherical symmetry, all plane symmetric electromagnetic fields obtained as solutions of Einstein-Maxwell equations are necessarily static.

STATIC PLANE SYMMETRIC SOLUTIONS OF THE EINSTEIN-MAXWELL EQUATIONS

From the discussion in Section III it follows that when both the metric tensor and the Maxwell field tensor admit plane symmetry, their nonvanishing components are given by

$$g_{11} = -g_{44} = -e^{2u} \quad , \qquad g_{22} = g_{33} = -e^{2v}$$

$$F_{14} \quad , \qquad F_{23} \tag{A.1}$$

where all the $g_{\mu\nu}$ and $F_{\mu\nu}$ depend on the variables x^1 and x^4 only. We first obtain the general solution of the eight Maxwell equations, the last two in the set of equations (3.1). This gives F_{14} and F_{23} in terms of the metric components u and v and certain other constants of integration. From the four equations, the third in the set of equations (3.1), we easily obtain, (taking account of the fact that $F_{\alpha\beta}$ has only two nonvanishing components F_{14}, F_{23} which depend on x^1 and x^4, and $F_{\alpha\beta} = -F_{\beta\alpha}$),

$$F_{23,1} = 0, \qquad F_{23,4} = 0 \quad , \tag{A.2}$$

hence, it follows that

$$F_{23} = -F_{32} = \beta \quad ,$$

where β is an arbitrary constant. This shows that magnetic field is constant both in space and time. The remaining four of the eight Maxwell equations give, assuming that $\det.\left|g_{\mu\nu}\right| \neq 0$, the following two equations

$$\left(F^{14} e^{2u} e^{2v}\right)_{,1} = 0$$

$$\left(F^{41} e^{2u} e^{2v}\right)_{,4} = 0 \quad . \tag{A.3}$$

Because $\quad F^{\alpha\beta} = F_{\rho\sigma} g^{\rho\alpha} g^{\sigma\beta}, \quad$ we have

$$F^{14} = -F_{14} e^{-4u} = -F^{41} \quad .$$

Then, it follows from the equations (A.3) that

$$F_{14} \, e^{-2u} \, e^{2v} \; = \; -\alpha \; = \; -F_{41} \, e^{-2u} \, e^{2v} \tag{A.4}$$

where α is an arbitrary constant. The non-vanishing components of the Maxwell stress energy tensor as calculated from the second in the set of equations (3.1) are given by:

$$T_1^1 \; = \; T_4^4 \; = \; -T_2^2 \; = \; -T_3^3 \; = \; \div \frac{1}{2} \, (\alpha^2 + \beta^2) e^{-4v}. \tag{A.5}$$

It can easily be verified, as it should be the case for a Maxwell energy tensor T^μ_ν, where the field tensor $F_{\alpha\beta}$ is obtained as a solution of the Maxwell equations, that

$$T \; \equiv \; T^\alpha_\alpha \; = \; 0$$

$$T^\beta_{\alpha;\beta} \; \equiv \; e^{-2u} \, e^{-2v} (e^{2u} \, e^{2v} \, T^\beta_\alpha)_{,\beta} \, - \, \frac{1}{2} \, g_{\beta\sigma,\alpha} \, T^{\beta\sigma} \; = \; 0 \quad .$$

COMPLETE SOLUTION OF EINSTEIN-MAXWELL EQUATIONS FOR STATIC PLANE SYMMETRIC FIELDS

The Einstein-Maxwell field equations, the first in the set (3.1) could be written as:

$$R^\mu_\nu \; = \; k T^\mu_\nu \tag{A.6}$$

because of the vanishing of the scalar invariant T for the Maxwell stress energy tensor. The components of the Ricci tensor for a plane symmetric space-time, i.e. when the metric tensor is given by (A.1), are given by Taub[7]. They are
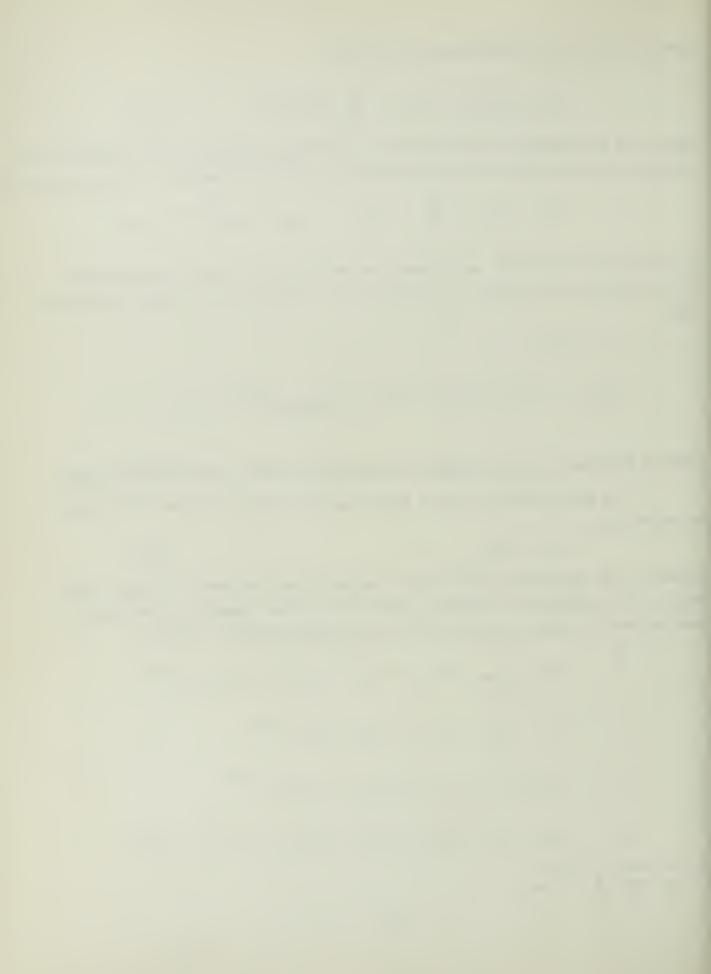
$$R_1^1 \; = \; -(u_{,11} - u_{,44} + 2v_{,11} + 2v_{,1}^2 - 2v_{,4} \, u_{,4} - 2v_{,1} \, u_{,1}) \, e^{-2u}$$

$$R_2^2 \; = \; R_3^3 \; = \; (v_{,44} + 2v_{,4}^2 - v_{,11} - 2v_{,1}^2) \, e^{-2u}$$

$$R_4^1 \; = \; -2(v_{,14} + v_{,4} \, v_{,1} - v_{,4} \, u_{,1} - v_{,1} \, u_{,4}) \, e^{-2u}$$

$$R_4^4 \; = \; (u_{,44} - u_{,11} + 2v_{,44} + 2v_{,4}^2 - 2v_{,4} \, u_{,4} - 2v_{,1} \, u_{,1}) \, e^{-2u} \, .$$

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7. op. cit., A. H. Taub

The field equations (A.6), then become , when u and v are functions of x^1 alone (for the static case) and the energy tensor given by (A.5) , the following:

$$u_{,11} + 2v_{,1}^2 + 2v_{,11} - 2v_{,1}\,u_{,1} = \frac{k}{2}(\alpha^2 + \beta^2)e^{-4v}\,e^{2u}$$

$$\frac{1}{2}\,e^{-2u}(v_{,11} + 2v_{,1}^2) = -\frac{k}{2}(\alpha^2 + \beta^2)\,e^{-4v}\,e^{2u} \qquad\qquad (A.7)$$

$$-u_{,11} - 2v_{,1}\,u_{,1} = -\frac{k}{2}(\alpha^2 + \beta^2)\,e^{-4v}\,e^{2u} \qquad .$$

In the above, there are three ordinary differential equations for the determination of two unknown functions u and v. That such a system of equations is consistent is a consequence of the Bianchi identities and the divergence relations for the energy-tensor. Professor Taub pointed out that the consistency of the set of equations (A.7) could be established directly as follows: Let

$$A = v_{,1}^2 + 2v_{,1}\,u_{,1} + \frac{k}{2}(\alpha^2 + \beta^2)\,e^{-4v}\,e^{2u}$$

$$B = v_{,11} + v_{,1}^2 - 2v_{,1}\,u_{,1}$$

$$C = u_{,11} + 2u_{,1}\,v_{,1} - \frac{k}{2}(\alpha^2 + \beta^2)\,e^{-4v}\,e^{2u} \qquad .$$
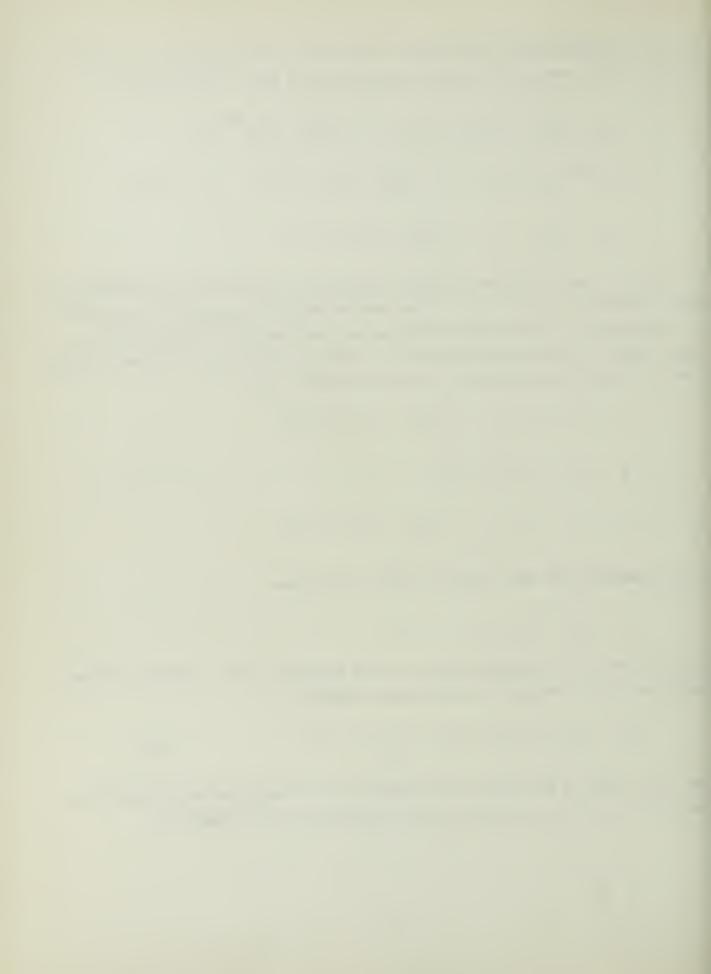
It can immediately be seen that the system of equations

$$A = 0, \quad B = 0, \quad C = 0$$

is equivalent to the original system of field equations (A.7). The above defined quantities A, B, C satisfy the differential equation

$$A_{,1} - 2(v_{,1} + u_{,1})(A + B) - 2v_{,1}\,C = 0 \qquad . \qquad\qquad (A.8)$$

That this indeed is an identity can easily be seen by substituting the expressions for A, B, C in it. The equation (3.10) is equivalent to the equations

$$\left(R^\mu_\nu - \frac{1}{2} g^\mu_\nu R - kT^\mu_\nu\right)_{;\mu} = 0 \; .$$

We first solve the two ordinary differential equations $A = 0$, $B = 0$, and determine the functions $u(x^1)$ and $v(x^1)$. These functions then must satisfy the third equation $C = 0$ as a consequence of the equation (A.8). The equation $B = 0$ is written as

$$\frac{v_{,11}}{v_{,1}} + v_{,1} - 2u_{,1} = 0$$

or

$$(\log v_{,1})_{,1} + (v - 2u)_{,1} = 0 \; .$$

On integration, we have

$$\log v_{,1} + v - 2u = a$$

where a is a constant of integration. The above equation gives

$$e^{2u} = b \, e^v \, v_{,1} \qquad\qquad (A.9)$$

where b is an arbitrary constant. Instead of the equation $B = 0$, we choose our second equation as $A + B = 0$, that is,

$$v_{,11} + 2v_{,1}^2 + \frac{k}{2}(\alpha^2 + \beta^2)\, e^{-4v}\, e^{2u} = 0$$
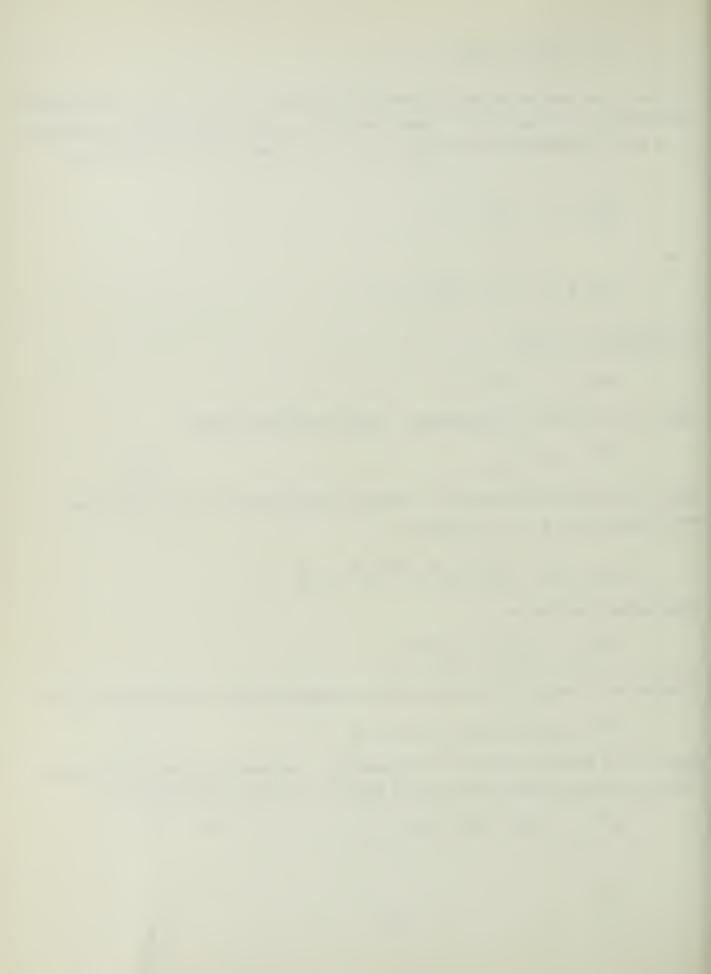
which may be rewritten as

$$(e^{2v})_{,11} - bk(\alpha^2 + \beta^2)(e^{-v})_{,1}$$

in which use is made of the equation (A.9). Integrating the above equation, we get

$$e^{2v}_{,1} - bk(\alpha^2 + \beta^2)(e^{-v}) + f = 0$$

where f is an arbitrary constant of integration. The above equation is transformed into the following with a change of the dependent variable defined by $e^v = P$,

$$2P^2\, P_{,1} - bk(\alpha^2 + \beta^2) + fP = 0 \; .$$

The above differential equation is integrated by separation of variables and one quadrature. The solution for P is implicitly given by the equation

$$\frac{P^2}{2} + \frac{bk}{f}(\alpha^2 + \beta^2)P + \frac{b^2k^2(\alpha^2 + \beta^2)}{f^2} \log[\, bk(\alpha^2 + \beta^2) - fP\,] = -\frac{f}{2}(x^1 + g)$$

where g is an arbitrary constant. Returning to the original variable v, we summarize the solution for Einstein-Maxwell equations for static plane symmetric case as:

$$ds^2 = -e^{2u(x^1)}[(dx^1)^2 - (dx^4)^2] - e^{2v(x^1)}\left[(dx^2)^2 + (dx^3)^2\right]$$

where u and v are given through the equations:

$$-fx^1 - fg = e^{2v} + \frac{2bk}{f}(\alpha^2 + \beta^2)\, e^v + \frac{2b^2k^2(\alpha^2 + \beta^2)}{f^2} \log[\, bk(\alpha^2 + \beta^2) - fe^v\,]$$

$$e^{2u} = be^v\, v_{,1} \qquad\qquad\qquad\qquad (A.10)$$

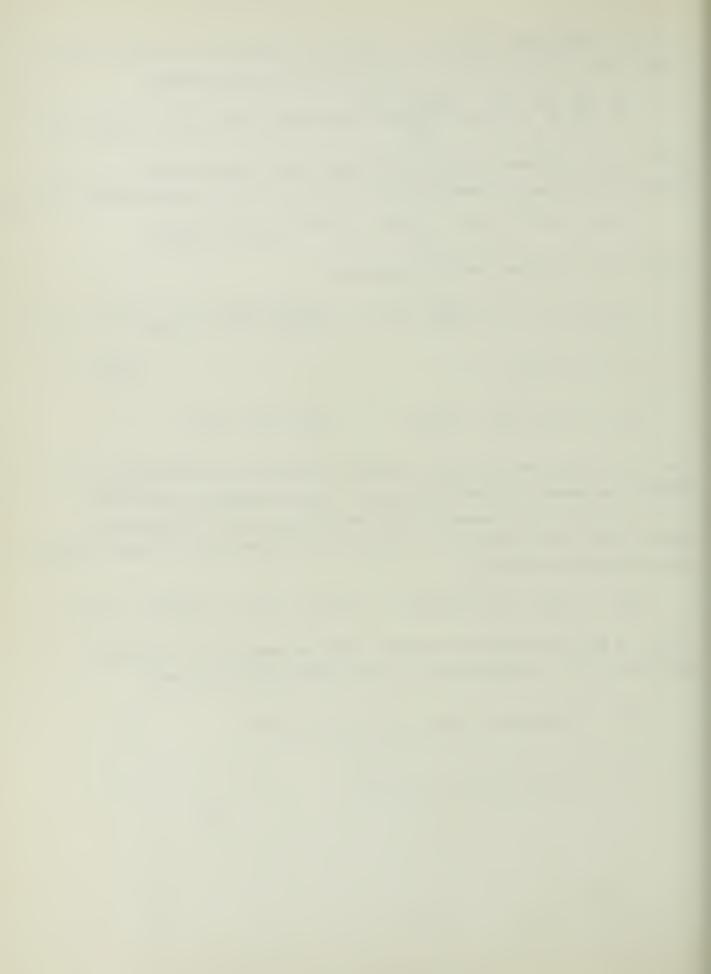$$F_{14} = -\alpha\, e^{2u}\, e^{-2v} = -F_{41}\,, \qquad\qquad F_{23} = \beta = -F_{32}$$

where b, f, g, α, β are arbitrary constants, the latter two characterizing the electric and magnetic fields, respectively. When one proceeds to the limit $\alpha \to 0$, $\beta \to 0$, the electromagnetic field vanishes, and as it is expected, the above solution (A.10) reduces to the solution of Professor Taub for plane symmetric empty space-times, that is,

$$ds^2 = -(fx^1 + g)^{-1/2}[(dx^1)^2 - (dx^4)^2] - (fx^1 + g)[(dx^2)^2 + (dx^3)^2]$$

where f, g are some arbitrary constants. When the constant $f \to 0$, then from equations (A.10), we have, when α, β do not vanish at the same time,

$$e^{2u} = \frac{b^2k}{2}(\alpha^2 + \beta^2)\,[\tfrac{3}{2}\, bk\,(\alpha^2 + \beta^2)\, x^1 + D]^{-2/3}$$

$$e^{2v} = [\tfrac{3}{2}\, bk\,(\alpha^2 + \beta^2)\, x^1 + D]^{2/3}$$

$$F_{14} = -F_{41} = -\alpha \frac{b^2 k}{2} (\alpha^2 + \beta^2) \left[\frac{3}{2} bk(\alpha^2 + \beta^2)x^1 + D\right]^{-4/3}$$

$$F_{23} = -F_{32} = \beta$$

where b, D, α, β are arbitrary constants. It may be remarked here that the solution given by (A.10) is the plane symmetric analogue of the Reissner-Nordstrom solution[8], that is,

$$ds^2 = -(1 - \frac{2m}{r} + \frac{\epsilon}{r^2})^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + (1 - \frac{2m}{r} + \frac{\epsilon}{r^2})dt^2$$

which generalized the Scwarzschild solution giving the exterior field of a spherical charged distribution of mass. Various time-dependent solutions for plane symmetric electromagnetic-graviational fields are found, and the question as to whether any of these gives a genuine non-static solution, that is, one for which transformations carrying it into the static solution (A.10) do not exist is being investigated at present. For a rigorous answer to this, groups of motion that such space-times admit are being obtained from solutions of Killing equations.

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8. op. cit., Sir Arthur Eddington